

Regularity of a class of differential operators.

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1 Introduction

In this paper we deal with the problem of regularity for non hypo-elliptic partial differential equations with polynomial coefficients. An operator $A : \mathcal{S}' \rightarrow \mathcal{S}'$ is regular if u is a Schwartz function whenever Au is a Schwartz function. It is well known that hypo-elliptic partial differential operators in the sense of Definition 25.2 of [8] are regular. On the other hand, the problem of finding necessary and sufficient conditions for the regularity of a differential operator with polynomial coefficients is still open.

In the case of ordinary differential equations, in [7] necessary and sufficient conditions for regularity are found under additional hypotheses. For partial differential equations the problem is much more complicated. We refer for example to [11], where the regularity of the twisted laplacian is proved, by explicit computation of the heat kernel and Green function. The twisted laplacian can be viewed as a Schrödinger operator perturbed with electromagnetic field; it is intimately connected with the sub-laplacian on the Heisenberg group, see [9], and is also studied from the point of view of spectral theory, see for example [6].

In this paper we follow a new approach, related to transformations of Wigner type, to prove regularity of a wide class of non hypo-elliptic partial differential operators with polynomial coefficients. The approach consists in applying a Wigner-like transform to a general partial differential equation. This idea is already present in some works related to engineering applications, cf. [2], [3]. In these papers some equations are analyzed, looking for the Wigner transform of the solution. Instead of finding first a solution u , and then computing its Wigner transform $\text{Wig}[u]$, the equation itself is Wigner-transformed obtaining an equation in $\text{Wig}[u]$. In this way it is possible to find, in some cases, the exact expression of $\text{Wig}[u]$.

In the present paper we use tensor products of topological vector spaces in order to apply the Wigner transform technique to the study of regularity in the Schwartz space of partial differential operators with polynomial coefficients. We refer to Section 3 for a precise statement of the result; the idea is that a linear (hypo-elliptic) operator is associated to a linear non hypo-elliptic one, and the Wigner-like transform allows to transfer the regularity from an operator to the other one. In this way we easily recover the regularity of the twisted laplacian, as well as of generalized versions of it. Moreover, we analyze several other examples, as general second order operators and operators associated to a complete Newton polygon.

In this paper, we consider only partial differential equations in two variables. There is no difficulty to generalize the results to an arbitrary even number of variables; we do not present our results in such a generality to keep the formalism as simple as possible.

The paper is organized as follows. In Section 2 we give general results concerning regularity and tensor product of operators; in Section 3 we prove the main result, and in Section 4 we apply such a result in concrete cases, by analyzing several examples.

2 Regularity and extension of the variables.

Given a linear operator A on $\mathcal{S}'(\mathbb{R})$ such that $A(\mathcal{S}(\mathbb{R})) \subset L^2(\mathbb{R})$, we denote by A^* the adjoint of $A|_{\mathcal{S}}$ with respect to the inner product on $L^2(\mathbb{R})$. The domain $\mathcal{D}(A^*)$ of A^* is given by all $u \in L^2(\mathbb{R})$ for which there exists $v \in L^2(\mathbb{R})$ such that

$$(A\phi | u)_{L^2} = (\phi | v)_{L^2}, \quad \phi \in \mathcal{S}.$$

Proposition 1. Consider a continuous linear operator A on $\mathcal{S}'(\mathbb{R})$. If $A(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$, then $\mathcal{S}(\mathbb{R}) \subset \mathcal{D}(A^*)$, A^{**} is the closure of $A|_{\mathcal{S}}$ in $L^2(\mathbb{R})$ and $A|_{\mathcal{S}}$ is continuous on $\mathcal{S}(\mathbb{R})$.

Proof. For all $\psi \in \mathcal{S}(\mathbb{R})$,

$$u \mapsto \langle Au | \overline{\psi} \rangle$$

is a continuous functional on $L^2(\mathbb{R})$. Then there exists $v \in L^2(\mathbb{R})$ such that

$$\langle Au | \overline{\psi} \rangle = (u | v)_{L^2}, \quad u \in L^2(\mathbb{R}).$$

This implies that $\psi \in \mathcal{D}(A^*)$ and that $A^*\psi = v$. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it follows that A^{**} is a closed extension of $A|_{\mathcal{S}}$.

Consider a sequence (u_n) in $\mathcal{S}(\mathbb{R})$ such that $u_n \rightarrow u$ and $Au_n \rightarrow v$ in $\mathcal{S}(\mathbb{R})$. Then $u_n \rightarrow u$ and $Au_n \rightarrow v$ in $\mathcal{S}'(\mathbb{R})$. But A is continuous on $\mathcal{S}'(\mathbb{R})$, hence $Au = v$. Thus A is a closed operator on $\mathcal{S}(\mathbb{R})$. Therefore A is continuous on $\mathcal{S}(\mathbb{R})$, by Closed Graph Theorem. \square

Definition 2. A linear operator A on $\mathcal{S}'(\mathbb{R})$ is *regular* if

$$Au \in \mathcal{S}(\mathbb{R}) \implies u \in \mathcal{S}(\mathbb{R}), \quad u \in \mathcal{S}'(\mathbb{R}).$$

We denote by $E \widehat{\otimes} F$ the *topological* tensor products of two nuclear spaces E and F . Given two linear continuous operators $A_j : E_j \rightarrow F_j$, with $j \in \{1, 2\}$, between nuclear spaces, $A_1 \widehat{\otimes} A_2 : E_1 \widehat{\otimes} E_2 \rightarrow F_1 \widehat{\otimes} F_2$ is the unique continuous linear operator such that $A_1 \otimes A_2(u_1 \otimes u_2) = A_1(u_1) \otimes A_2(u_2)$, for all $(u_1, u_2) \in E_1 \times E_2$.

Given a linear operator A on a vector space E , we denote by $\mathcal{N}(A)$ the subspace of the solutions to the equation $Au = 0$.

Theorem 3. Consider a continuous linear operator A on $\mathcal{S}'(\mathbb{R})$ such that $A(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$ and assume that

- (i) $\mathcal{N}(A^*) \subset \mathcal{S}(\mathbb{R})$,
- (ii) $\mathcal{S}(\mathbb{R}) = A(\mathcal{S}(\mathbb{R})) \oplus \mathcal{N}(A^*)$.
- (iii) $\mathcal{S}'(\mathbb{R}) = A(\mathcal{S}'(\mathbb{R})) \oplus \mathcal{N}(A^*)$,

Let I be the identity operator on $\mathcal{S}'(\mathbb{R})$. Then A is regular and one-to-one if and only if the tensor product $A \widehat{\otimes} I$ is regular on $\mathcal{S}'(\mathbb{R}^2)$.

Remark. We say that the operator $A \widehat{\otimes} I$ is obtained by “extension of the variables from \mathbb{R} to \mathbb{R}^2 ”. This explains the title of this section.

Proof. Assume $A \widehat{\otimes} I$ is regular.

Consider $u \in \mathcal{S}'(\mathbb{R})$ such that $Au \in \mathcal{S}(\mathbb{R})$. Then $A \widehat{\otimes} I(u \otimes v) = Au \otimes v \in \mathcal{S}(\mathbb{R}^2)$ for all $v \in \mathcal{S}(\mathbb{R})$. If $A \widehat{\otimes} I$ is regular, $u \otimes v$ must belong to $\mathcal{S}(\mathbb{R}^2)$ for all $v \in \mathcal{S}(\mathbb{R})$. But this is impossible, unless u belongs to $\mathcal{S}(\mathbb{R})$.

Assume now there exists $\phi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ such that $A\phi = 0$. Let δ be the Dirac distribution, then $\phi(x) \otimes \delta(y)$ belongs to the kernel of $A \widehat{\otimes} I$, but not to $\mathcal{S}(\mathbb{R}^2)$, in contradiction with the regularity of $A \widehat{\otimes} I$.

Assume now that A is regular and one-to-one. By assumption (ii) and Open Mapping Theorem $A|_{\mathcal{S}}$ is an isomorphism of $\mathcal{S}(\mathbb{R})$ onto $A(\mathcal{S}(\mathbb{R}))$. Then, by Propositions 43.7 and 43.9 of [10], $A|_{\mathcal{S}} \widehat{\otimes} I|_{\mathcal{S}}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^2)$ onto

$$A|_{\mathcal{S}} \widehat{\otimes} I|_{\mathcal{S}}(\mathcal{S}(\mathbb{R}^2)) = A(\mathcal{S}(\mathbb{R})) \widehat{\otimes} \mathcal{S}(\mathbb{R}).$$

Since

$$A|_{\mathcal{S}} \widehat{\otimes} I|_{\mathcal{S}} = (A \widehat{\otimes} I)|_{\mathcal{S}},$$

we have that $(A \widehat{\otimes} I)|_{\mathcal{S}}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^2)$ onto $A(\mathcal{S}(\mathbb{R})) \widehat{\otimes} \mathcal{S}(\mathbb{R})$.

Moreover, thanks to hypothesis (ii), we have:

$$(1) \quad \mathcal{S}(\mathbb{R}^2) = \left(A(\mathcal{S}(\mathbb{R})) \oplus \mathcal{N}(A^*) \right) \widehat{\otimes} \mathcal{S}(\mathbb{R}) = A \widehat{\otimes} I(\mathcal{S}(\mathbb{R}^2)) \oplus \mathcal{N}(A^*) \widehat{\otimes} \mathcal{S}(\mathbb{R}).$$

By Proposition 3.17.2 of [4], the Open Mapping Theorem is true for continuous linear maps from a Pták space onto a barrelled Hausdorff space. Now by Proposition 3.17.6 of [4], $\mathcal{S}'(\mathbb{R})$ is a Pták space. On the other hand, from hypothesis (iii) we have that $A(\mathcal{S}'(\mathbb{R}))$ is canonically isomorphic to $\mathcal{S}'(\mathbb{R})/\mathcal{N}(A^*)$. Then, by Corollary (a) to Proposition 3.6.4 of [4], we have that $A(\mathcal{S}'(\mathbb{R}))$ is a barrelled Hausdorff space. Then, by Open Mapping Theorem, A is an isomorphism of $\mathcal{S}'(\mathbb{R})$ onto $A(\mathcal{S}'(\mathbb{R}))$.

Therefore from Propositions 43.7 and 43.9 of [10] we obtain that $A \widehat{\otimes} I$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^2)$ onto a dense subspace of $A(\mathcal{S}'(\mathbb{R})) \widehat{\otimes} \mathcal{S}'(\mathbb{R})$. Since $\mathcal{S}'(\mathbb{R}^2)$ is complete, we have

$$A \widehat{\otimes} I(\mathcal{S}'(\mathbb{R}^2)) = A(\mathcal{S}'(\mathbb{R})) \widehat{\otimes} \mathcal{S}'(\mathbb{R}).$$

Moreover, since $\mathcal{S}'(\mathbb{R}) = A(\mathcal{S}'(\mathbb{R})) \oplus \mathcal{N}(A^*)$, we obtain

$$(2) \quad \mathcal{S}'(\mathbb{R}^2) = A(\mathcal{S}'(\mathbb{R})) \widehat{\otimes} \mathcal{S}'(\mathbb{R}) \oplus \mathcal{N}(A^*) \otimes \mathcal{S}'(\mathbb{R}) = A \widehat{\otimes} I(\mathcal{S}'(\mathbb{R}^2)) \oplus \mathcal{N}(A^*) \widehat{\otimes} \mathcal{S}'(\mathbb{R}).$$

Consider now $u \in \mathcal{S}'(\mathbb{R}^2)$ such that $f = (A \widehat{\otimes} I)u \in \mathcal{S}(\mathbb{R}^2)$. Since f belongs to $\mathcal{S}(\mathbb{R}^2)$, thanks to (1), there exist unique $v \in \mathcal{S}(\mathbb{R}^2)$ and $h \in \mathcal{N}(A^*) \widehat{\otimes} \mathcal{S}'(\mathbb{R}) \subset \mathcal{N}(A^*) \widehat{\otimes} \mathcal{S}'(\mathbb{R})$ such that $(A \widehat{\otimes} I)u = (A \widehat{\otimes} I)v + h$. But then, (2) implies that $h = 0$ and $u = v \in \mathcal{S}(\mathbb{R}^2)$. \square

3 Regularity of a class of differential operators.

Definition 4. A polynomial $a(x, \xi)$ on $\mathbb{R} \times \mathbb{R}$ is *hypo-elliptic* if it does not vanish outside a compact set and

$$\lim_{|x|+|\xi| \rightarrow \infty} \frac{|\partial_x a(x, \xi)| + |\partial_\xi a(x, \xi)|}{|a(x, \xi)|} = 0.$$

Theorem 5. A differential operator $A : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ with polynomial hypo-elliptic symbol is regular and satisfies the hypotheses of Theorem 3.

Proof. By Tarski-Seidenberg Theorem, see Appendix A of [5], we have that a is hypo-elliptic in the sense of Definition 25.2 of [8]. Then the result follows from Theorem 25.3 of [8]. \square

Theorem 6. Consider a differential operator $A : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ with polynomial hypo-elliptic symbol. Then $A \widehat{\otimes} I$ is regular if and only if A is one-to-one.

Proof. It follows from Theorems 3 and 5. \square

Theorem 7. Consider the differential operator on $\mathcal{S}'(\mathbb{R}^2)$

$$(3) \quad B = \sum_{j+k \leq m} c_{j,k}(x - qD_y)^j (y + pD_x)^k,$$

where p is a real number and

$$q = 1 - p.$$

Let A be the differential operator on $\mathcal{S}'(\mathbb{R})$ with symbol

$$a(x, \xi) = \sum_{j+k \leq m} c_{j,k} x^j \xi^k.$$

Assume the symbol a be hypo-elliptic. Then B is regular if and only if A is one-to-one.

Proof. Introduce the Wigner-like transform of a function $f \in \mathcal{S}(\mathbb{R}^2)$:

$$\text{Wig}_p[f](x, y) = \frac{1}{(2\pi)^{\nu/2}} \int e^{-izy} f(x + (1-p)z, x - pz) dz.$$

Since Wig_p an isomorphism both on $\mathcal{S}(\mathbb{R}^2)$ and on $\mathcal{S}'(\mathbb{R}^2)$, the result follows from Theorem 6 and the following identity.

$$(4) \quad B \text{Wig}_p[w] = \text{Wig}_p((A \widehat{\otimes} I)w), \quad w \in \mathcal{S}(\mathbb{R}^2).$$

We prove (4), by induction. This means that we may assume $m = 1$.

Define the operators:

$$M_1 w(x, y) = xw(x, y), \quad M_2 w(x, y) = yw(x, y),$$

and

$$D_1 w(x, y) = D_x w(x, y), \quad D_2 w(x, y) = D_y w(x, y),$$

where, as usual,

$$D_x = -i\partial_x, \quad D_y = -i\partial_y.$$

Then we have

$$(5) \quad \begin{aligned} D_1 \text{Wig}_p[w](x, y) &= \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} D_x (w(x + qz, x - pz)) dz \\ &= \text{Wig}_p[D_1 w](x, y) + \text{Wig}_p[D_2 w](x, y), \end{aligned}$$

$$(6) \quad \begin{aligned} D_2 \text{Wig}_p[w](x, y) &= \frac{1}{(2\pi)^{\nu/2}} \int -z_j e^{-iyz} w(x + qz, x - pz) dz \\ &= \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} (x - pz) w(x + qz, x - pz) dz \\ &\quad - \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} (x + qz) w(x + qz, x - pz) dz \\ &= \text{Wig}_p[M_2 w](x, y) - \text{Wig}_p[M_1 w](x, y), \end{aligned}$$

$$(7) \quad \begin{aligned} M_1 \text{Wig}_p[w](x, y) &= \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} xw(x + qz, x - pz) dz \\ &= \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} qxw(x + qz, x - pz) dz \\ &\quad + \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} pxw(x + qz, x - pz) dz \\ &= \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} q(x - pz)w(x + qz, x - pz) dz \\ &\quad + \frac{1}{(2\pi)^{\nu/2}} \int e^{-iyz} p(x + qz)w(x + qz, x - pz) dz \\ &= q \text{Wig}_p[M_2 w](x, y) + p \text{Wig}_p[M_1 w](x, y), \end{aligned}$$

$$\begin{aligned}
(8) \quad M_2 \text{Wig}_p[w](x, y) &= \frac{1}{(2\pi)^{\nu/2}} \int \left(-D_z e^{-iyz} \right) w(x + qz, x - pz) dz \\
&= \frac{1}{(2\pi)^{\nu/2}} q \int e^{-iyz} (D_1 w)(x + qz, x - pz) dz \\
&\quad - \frac{1}{(2\pi)^{\nu/2}} p \int e^{-iyz} (D_2 w)(x + qz, x - pz) dz \\
&= q \text{Wig}_p[D_1 w](x, y) - p \text{Wig}_p[D_2 w](x, y).
\end{aligned}$$

Then from (5) and (8) we obtain:

$$\text{Wig}_p[D_1 w] = (M_2 + pD_1) \text{Wig}_p[w],$$

and from (6) and (7) we obtain:

$$\text{Wig}_p[M_1 w] = (M_1 - qD_2) \text{Wig}_p[w]. \quad \square$$

Observe that the symbol of the operator (3) cannot be hypo-elliptic in the sense of the Definition 25.2 of [8]. In fact we have the following proposition.

Proposition 8. Let $b(x, y; \xi, \eta)$ be the symbol of the operator (3), and

$$\tilde{a}(x, \xi) = \sum_{j+k \leq m} \sum_{n \leq \max\{j, k\}} c_{j, k} (iq)^n n! \binom{j}{n} \binom{k}{n} x^{j-n} \xi^{k-n}.$$

Then for all $(x_0, \xi_0) \in \mathbb{R} \times \mathbb{R}$, we have that

$$b(x_0 + q\eta, y_0 - p\xi; \xi, \eta) = \tilde{a}(x_0, y_0), \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}.$$

Proof. By Theorem 3.4 of [8] we have that the symbol of the operator B is given by

$$\begin{aligned}
b(x, y; \xi, \eta) &= \sum_{j+k \leq m} c_{j, k} \sum_{n \in \mathbb{Z}_+} \frac{(-i)^n}{n!} \partial_\eta^n (x - q\eta)^j \partial_y^n (y + p\xi)^k \\
&= \sum_{j+k \leq m} c_{j, k} \sum_{n \leq \max\{j, k\}} (iq)^n \binom{j}{n} \binom{k}{n} n! (x - \eta + p\eta)^{j-n} (y + p\xi)^{k-n}. \quad \square
\end{aligned}$$

We end this section with a simple observation which widens the range of applicability of the Theorem 7.

Proposition 9. Given a differential operator $B = b(z, D)$ on \mathbb{R}^2 , with polynomial coefficients, consider a 2×2 non-singular real matrix T and set

$$B_T = a_T(z, D) = a(T'z, T^{-1}D).$$

where T' is the transposed matrix. Then B is regular if and only if B_T is regular.

Proof. Consider $u \in \mathcal{S}'(\mathbb{R}^2)$. A simple computation yields

$$(9) \quad B_T(u \circ T') = (Bu) \circ T'.$$

Since $u \circ T'$ belongs to $\mathcal{S}(\mathbb{R}^2)$ if and only if $u \in \mathcal{S}(\mathbb{R}^2)$, the result follows immediately from (9). \square

4 Examples.

4.1 Second order self-adjoint operators.

Proposition 10. Consider the polynomial

$$(10) \quad a(x, \xi) = a_2 x^2 + a_1 x + a_0 - i b_1 + 2(b_1 x + b_0)\xi + c_0 \xi^2,$$

where the coefficients $a_2, a_1, a_0, b_1, b_0, c_0$ are real numbers and $i^2 = -1$.

Assume there exist r_1, r_0, s_1, s_0 such that

$$(11) \quad b_1 = r_1 s_1, \quad b_0 = r_0 s_0$$

and

$$(12) \quad s_0^2 + s_1^2 \leq c_0, \quad a_1^2 < 4(a_2 - r_1^2)(a_0 - r_0^2).$$

Then the operator

$$B = a_2(x - qD_y)^2 + a_1(x - qD_y) + a_0 - ib_1 + 2(b_1(x - qD_y) + b_0)(y + pD_x) + c_0(y + pD_x)^2$$

is regular, for all $p \in \mathbb{R}$ and $q = 1 - p$.

Proof. From (11) and (12) we obtain that

$$b_1^2 = (r_1 s_1)^2 < a_2 c_0,$$

therefore the quadratic form

$$a_2 x^2 + 2b_1 x \xi + c_0 \xi^2$$

is positive-definite and this implies that the polynomial a is hypo-elliptic.

A simple computation shows that the operator $A = a(x, D)$ is symmetric on $\mathcal{S}(\mathbb{R})$. This implies that $(Au | u)$ is real for all $u \in \mathcal{S}(\mathbb{R})$. Then, by using (11) and (12), we have the estimate

$$(13) \quad \begin{aligned} (Au | u)_{L^2} &= \int \left((a_2 x^2 + a_1 x + a_0) |u|^2 - ib_1 |u|^2 + 2(b_1 x + b_0) Du \bar{u} + c_0 D^2 u \bar{u} \right) dx \\ &= \int \left((a_2 x^2 + a_1 x + a_0) |u|^2 + (b_1 x + b_0) (Du \bar{u} + u \overline{Du}) + c_0 |Du|^2 \right) dx \\ &\geq \int \left(((a_2 - r_1^2)x^2 + a_1 x + a_0 - r_0^2) |u|^2 + (c_0 - s_1^2 - s_0^2) |Du|^2 \right) dx \\ &\geq \frac{4(a_0 - r_0^2)(a_2 - r_1^2) - a_1^2}{a_2 - r_1^2} \|u\|_{L^2}^2. \end{aligned}$$

Now, if $Au = 0$, by regularity u belongs to $\mathcal{S}(\mathbb{R})$, and estimate (13) together with (12) implies $u = 0$. This shows that A is one-to-one. Then the result follows from Theorem 7. \square

In particular, when $a_2 = 4$, $c_0 = 1/4$, $a_1 = a_0 = b_1 = b_0 = 0$ and $p = q = 1/2$, Proposition 10 implies the regularity of

$$(14) \quad B = 4\left(x - \frac{1}{2}D_y\right)^2 + \frac{1}{4}\left(y + \frac{1}{2}D_x\right)^2.$$

Then Proposition 9 with

$$T = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix},$$

implies the regularity of the twisted laplacian, already studied in [11]:

$$(15) \quad B_T = \left(D_y - \frac{1}{2}x\right)^2 + \left(D_x + \frac{1}{2}y\right)^2$$

In the next section we study some generalizations of this operator.

4.2 Generalized twisted laplacian.

Proposition 11. Fix positive integers h, k, m and n , with

$$(16) \quad m < h, \quad n < k, \quad nh + mk \geq hk.$$

Let $\lambda, \sigma > 0$, $\mu, \nu \geq 0$, $p \in \mathbb{R}$, and write $q = 1 - p$. Then the operator

$$(17) \quad B = \lambda(x - qD_y)^{2h} + \mu(x - qD_y)^m(y + pD_x)^{2n}(x - qD_y)^m \\ + \nu(y + pD_x)^n(x - qD_y)^{2m}(y + pD_x)^n + \sigma(y + pD_x)^{2k}$$

is regular.

Proof. Consider the operator A on $\mathcal{S}'(\mathbb{R})$ defined by

$$A = \lambda x^{2h} + \mu x^m D^{2n} x^m + \nu D^n x^{2m} D^n + \sigma D^{2k},$$

and let $a(x, \xi)$ be its symbol. Conditions (16) imply that $a(x, \xi)$ is multi-quasi-elliptic, in the sense of definition of page 62 of [1]. As a matter of facts, an easy computation shows that $a(x, \xi)$ is hypo-elliptic.

Let us prove that $A = a(x, D)$ is one-to-one. Given $u \in \mathcal{S}(\mathbb{R})$ we have

$$(Au | u)_{L^2} = \lambda(M^{2h}u | u)_{L^2} + \mu(M^m D^{2n} M^m u | u)_{L^2} + \nu(D^n M^{2m} D^n u | u)_{L^2} + \sigma(D^{2k}u | u)_{L^2} \\ = \lambda \|M^h u\|_{L^2}^2 + \mu \|D^n M^m u\|_{L^2}^2 + \nu \|M^m D^n u\|_{L^2}^2 + \sigma \|D^k u\|_{L^2}^2 \geq 0,$$

where $Mu(x) = xu(x)$ is the multiplication operator. Then, $Au = 0$ implies $M^h u = 0$ and $D^k u = 0$, that is $u = 0$. So A is one-to-one, and from Theorem 7 the operator B is regular. \square

Observe that Proposition 8 implies that the symbol of the operator (17) is not hypo-elliptic.

When $h = k = 1$, $\lambda = 4$, $\sigma = 1/4$, $\mu = \nu = 0$ and $q = p = 1/2$, we recover the regularity of (14), and consequently of (15), as in Section 4.1. On the other hand, we can now treat a quasi homogeneous twisted laplacian of higher order, with arbitrary coefficients. Consider $\rho, \tau \in \mathbb{R}$ such that

$$\rho\tau \neq 0, \quad \rho \neq \tau,$$

and let B the operator (17) with $\mu = \nu = 0$, $\lambda = (\rho - \tau)^{2h}$, $\sigma = 1$, $p = \frac{\rho}{\rho - \tau}$. Consider moreover

$$T = \begin{bmatrix} \frac{\rho}{\rho - \tau} & 0 \\ 0 & \tau \end{bmatrix}.$$

Then by Proposition 11 and Proposition 9 we have that the operator

$$(18) \quad B_T = (D_y + \rho x)^{2h} + (D_x + \tau y)^{2k}$$

is regular.

4.3 Weyl-Wick transform and positivity of operators.

The problem of proving injectivity of an operator is in general non-trivial, since it is strictly connected to the knowledge of its spectrum. On the other hand the (strict) positivity is a sufficient condition for an operator to be one-to-one. In this section we give a general method to construct hypo-elliptic polynomial symbols of positive operators. Applying Theorem 7 we then obtain regular operators with non hypo-elliptic symbol.

Definition 12. The *Weyl-Wick transform* of the polynomial

$$(19) \quad a(x, \xi) = \sum_{j+k \leq m} c_{j,k} x^j \xi^k, \quad (x, \xi) \in \mathbb{R} \times \mathbb{R},$$

is the polynomial

$$(20) \quad W[a](x, \xi) = \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{4^n n!} \Delta_{x, \xi}^n \sum_{l \in \mathbb{Z}_+} \frac{(-i)^l}{2^l l!} \partial_x^l \partial_\xi^l a(x, \xi).$$

Proposition 13. The Weyl-Wick transform of a polynomial $a(x, \xi)$ of order m is a polynomial of order m and W is invertible on the space of polynomials in the (x, ξ) variables; in particular

$$W^{-1}[a](x, \xi) = \sum_{l \in \mathbb{Z}_+} \frac{i^l}{2^l l!} \partial_x^l \partial_\xi^l \sum_{n \in \mathbb{Z}_+} \frac{1}{4^n n!} \Delta_{x, \xi}^n a(x, \xi),$$

Proof. The simple proof is left for the reader, see Problem 24.11 and Theorem 23.3 of [8]. \square

Consider now the function

$$\Phi_{y, \eta}(x) = \pi^{-1/4} e^{ix\eta} e^{-\frac{1}{2}(y-x)^2},$$

where $y, \eta \in \mathbb{R}$ are parameters, and the corresponding orthogonal projection in $L^2(\mathbb{R})$ on $\Phi_{y, \eta}$:

$$(21) \quad P_{y, \eta} u(x) = \left(\int u(t) \overline{\Phi_{y, \eta}(t)} dt \right) \Phi_{y, \eta}(x),$$

for $u \in L^2(\mathbb{R})$. We have the following result.

Proposition 14. Let A be the differential operator with symbol (19), then we have

$$(22) \quad Au = \frac{1}{2\pi} \int W[a](y, \eta) (P_{y, \eta} u) dy d\eta, \quad u \in \mathcal{S}(\mathbb{R}).$$

Proof. It follows from Problem 24.10 and Theorem 23.3 of [8]. \square

Proposition 15. If a is a polynomial such that $W[a](x, \xi) > 0$ for almost all $(x, \xi) \in \mathbb{R}^2$, then the operator $A = a(x, D)$ is one-to-one.

Proof. It follows from (22) and (21) that

$$(Au | u)_{L^2} = \frac{1}{2\pi} \int W[a](y, \eta) \left| \int u(x) \overline{\Phi_{y, \eta}(x)} dx \right|^2 dy d\eta.$$

Then, if $W[a] > 0$ almost everywhere, $Au = 0$ implies that

$$\pi^{-1/4} \int e^{-ix\eta} e^{-\frac{1}{2}(y-x)^2} u(x) dx = \int u(x) \overline{\Phi_{y, \eta}(x)} dx = 0, \quad (y, \eta) \in \mathbb{R} \times \mathbb{R},$$

and therefore

$$e^{-\frac{1}{2}(y-x)^2} u(x) = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R},$$

that is $u = 0$. \square

Since the Weyl-Wick transform preserves hypo-ellipticity, we have the following result.

Corollary 16. Let $a(x, \xi)$ be an hypo-elliptic polynomial such that $a(x, \xi) > 0$ for almost all $(x, \xi) \in \mathbb{R} \times \mathbb{R}$, and let $r(x, \xi) = W^{-1}[a](x, \xi)$. Write

$$r(x, \xi) = \sum_{j+k \leq m} c_{j,k} x^j \xi^k,$$

where $m \in \mathbb{Z}_+$ and $c_{j,k} \in \mathbb{R}$, $j+k \leq m$ are uniquely determined by a . Then the operator

$$B = \sum_{j+k \leq m} c_{j,k} (x - qD_y)^j (y + pD_x)^k,$$

with $q = 1 - p$, is regular.

4.4 Operators with negative index.

Now we consider differential operators of first order.

Proposition 17. Fix $\alpha \in \mathbb{C}$ and a positive integer m . Suppose that $(\operatorname{Im} \alpha)^m > 0$. Then for every $p \in \mathbb{R}$ the operator in \mathbb{R}^2

$$y + pD_x + \alpha(x + (p-1)D_y)^m$$

is regular.

Proof. Consider the operator A with symbol

$$(23) \quad a(x, \xi) = \xi + \alpha x^m.$$

Thanks to Theorem 7, it is enough to prove that a is hypo-elliptic and A is one-to-one. The condition $(\operatorname{Im} \alpha)^m > 0$ implies the hypo-ellipticity of a .

From Theorem 5 the kernel of A is then contained in $\mathcal{S}(\mathbb{R})$. On the other hand the classical solutions of $Au = 0$ are the functions

$$u(x) = \beta e^{-i\alpha \frac{x^{m+1}}{m+1}}, \quad \beta \in \mathbb{C},$$

which do not belong to $\mathcal{S}(\mathbb{R})$ for $(\operatorname{Im} \alpha)^m > 0$. This implies that A is one-to-one, and the proof is complete. \square

In view of the hypotheses of Theorem 3, it is interesting to observe that when m is odd and $\operatorname{Im} \alpha > 0$ is positive, the operator $A = D + \alpha x^m$ has index -1 , hence in particular $\mathcal{N}(A^*) \neq 0$.

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